AN ALGORITHM TO COMPUTE GENERALIZED PADÉ-HERMITE FORMS

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Abstract

An algorithm is given to compute the Padé-Hermite form for a vector of power series. The complexity of computing Padé-Hermite forms of type $(d_1, d_2, \ldots, d_n)$ is $O(n(d_1 + d_2 + \ldots + d_n)^2)$ which is the same as other fast algorithms. The advantage of this algorithm is that it also treats non-normal cases without any loss of speed. The algorithm also can be used to find a $k(z)$-linear between a sequence of power series in z. A very similar algorithm is found by Beckermann and Labahn independently (cf. [1, 2]).

1 Introduction

Suppose that $f(z) \in k[[z]]$ is a formal power series with coefficients in the field k. The pair $(P(z), Q(z))$, where P and Q are polynomials in z whose degrees are bounded by m and n respectively, is called Padé approximant of f if

$$Q(z)f(z) = P(z) + O(z^{m+n+1})$$

This means that $P(z)/Q(z)$ is a good approximation for $f(z)$. This definition is a special case of the definition of the Padé-Hermite approximant. Suppose $f_1(z), f_2(z), \ldots, f_n(z) \in k[[z]]$ are all power series. A vector $(P_1(z), P_2(z), \ldots, P_n(z))$ is called Padé-Hermite approximant of type $(d_1, d_2, \ldots, d_n)$ for $(f_1, f_2, \ldots, f_n)$ if

$$P_1(z)f_1(z) + P_2(z)f_2(z) + \ldots + P_n(z)f_n(z) = O(z^{d_1+d_2+\ldots+d_n+n-1}) \qquad (1)$$

and $\deg(P_i(z)) \leq d_i$ for all i. If we take $n = 2$ and $f_2(z) = -1$ then (P_1, P_2) is exactly the Padé-approximant of type (d_1, d_2) for the power series $f_1(z)$. To compute $P_1, P_2, \ldots, P_n$ one could proceed as follows: Write $P_i = \sum_{j=0}^{d_i} a_{i,j} z^j$ for all i. Equation (1) defines $d_1 + d_2 + \ldots + d_n + n - 1$ homogeneous linear equations in the variables $a_{i,j}$ $(1 \leq i \leq n,\ 0 \leq j \leq d_i)$. Because there are $d_1+d_2+\ldots+d_n+n$ variables there is a non-zero solution. Solving the system of linear equations costs $O((d_1+d_2+\ldots+d_n)^3)$ operations (multiplication, subtraction, etc.) in the field k. This can be done better.

Della Dora and Dicrescenzo gave in [6] an efficient algorithm to compute these Padé-Hermite approximants using Padé-Hermite tables. Further Paszkowski gave in [8] another efficient algorithm. Unfortunately these algorithms work only in the case that $(f_1, f_2, \ldots, f_n)$ is normal. Roughly speaking this means that (1) has for every type $(d_1, d_2, \ldots, d_n)$ only one solution $(P_1, P_2, \ldots, P_n)$ up to scalar multiplication with elements of the field k. In [4] Cabay and Labahn present an algorithm which also works in the non-normal case. But in the non-normal case the complexity might be worse than $O(n(d_1+d_2+\ldots+d_n)^2)$. In this paper an algorithm is presented which reaches $O(n(d_1+d_2+\ldots+d_n)^2)$ in the normal as well in the non-normal case.

2 Definitions and preliminaries

Definition 1 *A polynomial vector* $(P_1, P_2, \ldots, P_n) \in k[z]^n$ *is called a Padé-Hermite form of type* $(d_1, d_2, \ldots, d_n)$ *for the vector* $(f_1, f_2, \ldots, f_n) \in k[[z]]^n$ *of formal power series if*

1. $\deg(P_i) \leq d_i$ *for all* $i = 1, 2, \ldots, n$.
2. $\sum_{i=1}^n P_i f_i = O(z^{d_1+d_2+\ldots+d_n+n-1})$
3. $P_i \neq 0$ *for some* i.

Theorem 2 *For every power series vector* $(f_1, f_2, \ldots, f_n)$ *and every sequence* $d_1, d_2, \ldots, d_n$ *of non-negative integers, there exists a Padé-Hermite form of type* $(d_1, d_2, \ldots, d_n)$ *for* $(f_1, f_2, \ldots, f_n) \in k[[z]]^n$.

Proof: Represent the coefficients of P_i by indeterminates, say $P_i = \sum_{j=0}^{d_i} a_{i,j} z^j$ for all i. The second condition yields $d_1+d_2+\ldots+d_n+n-1$ linear equations, but there are $d_1+d_2+\ldots+d_n+n$ indeterminates. Hence there is a non-zero solution for the $a_{i,j}$. □

If $\vec{P} = (P_1, \ldots, P_n)$ is a polynomial vector, and $\vec{f} = (f_1, \ldots, f_n)$ is a vector of power series, then the inproduct $\sum_{i=1}^n P_i f_i \in k[[z]]$ is denoted by $\langle \vec{P}, \vec{f} \rangle$. If $g = \sum_{i=0}^{\infty} g_i z^i \in k[[z]]$ is a power series, then the valuation of g is defined by

$v(g) = \min\{i \mid g_i \neq 0\}$. If $g = 0$ then $v(g) = \infty$. For any polynomial vector $\vec{P} = (P_1, \ldots, P_n) \in k[z]^n$ we define $\deg(\vec{P}) = \max(\deg(P_1), \ldots, \deg(P_n))$. We will say that $\vec{P}$ is of type i, if $\deg(P_i) = \deg(\vec{P})$ and $\deg(P_j) < \deg(\vec{P})$ for all $j > i$. This will be denoted by $\mathrm{type}(\vec{P}) = i$. If $\vec{P}$ and $\vec{Q}$ are two polynomial vectors, then we will say that $\vec{P}$ is smaller than $\vec{Q}$ ($\vec{P} < \vec{Q}$) if $\deg(\vec{P}) < \deg(\vec{Q})$ or $\deg(\vec{P}) = \deg(\vec{Q})$ and $\mathrm{type}(\vec{P}) < \mathrm{type}(\vec{Q})$.

Theorem 3 *Properties of* type *and* deg*:*

1. *if* $\vec{P}$ *and* $\vec{Q}$ *are polynomial vectors, then* $\mathrm{type}(\vec{P} + \vec{Q}) \in \{\mathrm{type}(\vec{P}), \mathrm{type}(\vec{Q})\}$
2. *if* $\mathrm{type}(\vec{P}) = \mathrm{type}(\vec{Q})$ *and* $\vec{P} > \vec{Q}$ *then there exists a polynomial* $q \in k[z]$ *such that* $\vec{P} - q\vec{Q} < \vec{P}$
3. *if* $\vec{Q} < \vec{P}$ *then* $\deg(\vec{P} + \vec{Q}) = \deg(\vec{P})$ *and* $\mathrm{type}(\vec{P} + \vec{Q}) = \mathrm{type}(\vec{P})$

The proof is left to the reader.

Definition 4 *Let $V \subseteq k[z]^n$ be a free sub-module of rank n. A sequence of polynomial vectors $\vec{Q}_1, \vec{Q}_2, \ldots, \vec{Q}_n \in V$ is called minimal vector sequence for V if $\vec{Q}_i$ is a non-trivial polynomial vector with minimal degree of type i for $i = 1, 2, \ldots, n$.*

Observe that any free sub-module $V \subseteq k[z]^n$ of rank n has such a minimal vector sequence.

Theorem 5 *If $V, \vec{Q}_1, \ldots, \vec{Q}_n$ are as in the previous definition, then V is generated by $\vec{Q}_1, \ldots, \vec{Q}_n$ as a $k[z]$-module.*

Proof: Define $W = k[z]\vec{Q}_1 + k[z]\vec{Q}_2 + \ldots + k[z]\vec{Q}_n \subseteq V$. Suppose $V \neq W$. Let $\vec{P}$ be a polynomial vector in $V \setminus W$ which is minimal (with respect to the ordering on polynomial vectors). Let $i = \mathrm{type}(\vec{P})$. There exists a polynomial $q \in k[z]$ such that $\vec{P} - q\vec{Q}_i < \vec{P}$ (see theorem 3, property 2). Because of the minimality of $\vec{P}$ we get $\vec{P} - q\vec{Q}_i \in W$, hence $\vec{P} \in W$. Contradiction, so $V = W$. □

Now we fix a power series vector $\vec{f} = (f_1, \ldots, f_n) \in k[z]^n$ and we fix a free sub-$k[z]$-module $V \subseteq k[z]^n$ of rank n. For every j we define a sub-module of V by

$$V_j = \{\vec{P} \in V \mid v(\langle \vec{P}, \vec{f} \rangle) \geq j\}$$

Of course $V \subseteq V_j \subseteq z^j V$, therefore V_j is a free sub-module of V of rank n. Suppose that $\vec{Q}_1, \ldots, \vec{Q}_n$ is a minimal vector sequence for V_j. How to construct a minimal vector sequence for V_{j+1}? This can be done as follows: If $v(\langle \vec{Q}_i, \vec{f} \rangle) \geq j+1$ for all i, then $V_j = V_{j+1}$ and we are done. Otherwise choose an i such that

(a) $v(\langle \vec{Q}_i, \vec{f} \rangle) = j$

(b) if $v(\langle \vec{Q}_l, \vec{f} \rangle) = j$ for some $l \neq i$, then $\vec{Q}_i < \vec{Q}_l$

Suppose that $v(\langle \vec{Q}_l, \vec{f} \rangle) = j$ for some l. We can choose now some $\lambda \in k$ such that $v(\langle \vec{Q}_l - \lambda \vec{Q}_i, \vec{f} \rangle) = v(\langle \vec{Q}_l, \vec{f} \rangle - \lambda \langle \vec{Q}_i, \vec{f} \rangle) > j$. Replace $\vec{Q}_l$ by $\vec{Q}_l - \lambda \vec{Q}_i$. The degree and type of $\vec{Q}_l$ doesn't change (see theorem 3, property 3). So $\vec{Q}_1, \ldots, \vec{Q}_n$ remains a minimal vector sequence. After some wiping, we may assume that $v(\langle \vec{Q}_i, \vec{f} \rangle) = j$ and $v(\langle \vec{Q}_l, \vec{f} \rangle) > j$ for $l \neq i$. We will prove that $\vec{Q}_1, \ldots, \vec{Q}_{i-1}, z\vec{Q}_i, \vec{Q}_{i+1}, \ldots, \vec{Q}_n$ is a minimal vector sequence. If $\vec{P} \in V_{j+1}$ has type $l \neq i$, then $\vec{P} \in V_i$, hence $\deg(\vec{P}) \geq \deg(\vec{Q}_l)$. Suppose $\vec{P} \in V_{j+1}$ has type i. Write $\vec{P} = \sum_{l=1}^{n} a_l \vec{Q}_l$. If $a_i = 0$ then the type of $\vec{P}$ cannot be i (see theorem 3, property 1). If $a_i \neq 0$, then $v(a_i) > 0$ because $v(\langle \vec{P}, \vec{f} \rangle) \geq j+1$. Hence the degree of a_i is at least 1. So $\deg(\vec{P}) \geq \deg(a_i)\deg(\vec{Q}_i) \geq \deg(z\vec{Q}_i)$.

Let d be fixed non-negative integer. A Padé-Hermite form of type $(d, d, d, \ldots, d)$ for $\vec{f}$ is a polynomial vector $\vec{P} \in V_{nd+n-1}$ such that $\deg(\vec{P}) \leq d$. We know that such a $\vec{P}$ exists (see theorem 2). Assume that $\vec{Q}_1, \ldots, \vec{Q}_n$ is a minimal vector sequence of V_{nd+n-1} and choose an l such that $\deg(\vec{Q}_l)$ is minimal. Let $i = \text{type}(\vec{P})$. Now $\deg(\vec{Q}_l) \leq \deg(\vec{Q}_i) \leq \deg(\vec{P}) \leq d$. Therefore $\vec{Q}_l$ is also a Padé-Hermite form of type $(d, d, d, \ldots, d)$ for $\vec{f}$. This gives us the following algorithm:

3 An algorithm to compute a Padé-Hermite form of type $(d, d, d, \ldots, d)$

The algorithm below computes for a given vector $\vec{f}$ of formal power series in z the Padé-Hermite form of type $(d, d, d, \ldots, d)$. In the algorithm "coeff" stands for a procedure such that for any power series $f \in k[[z]]$ and any integer j the output of coeff(f, j) is the coefficient of z^j.

input: $\vec{f} = (f_1, f_2, \ldots, f_n)$, d
for k from 1 to n do
  $\vec{Q}_k := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ (n-dimensional vector with 1 as k-th entry)
for j from 0 to $nd + n - 2$ do
  $i := 0$
  for k from 1 to n do
    if $v(\langle \vec{Q}_i, \vec{f} \rangle) = j$ and ($\vec{Q}_k < \vec{Q}_i$ or $i = 0$) then $i := k$
  if $i \neq 0$ then
    $\mu := \text{coeff}(\langle \vec{Q}_i, \vec{f} \rangle, j)$
    $\vec{Q}_i := \mu^{-1} \vec{Q}_i$
    for k from 1 to n do
      if $k \neq i$ then
        $\lambda := \text{coeff}(\langle \vec{Q}_k, \vec{f} \rangle, j)$
        $\vec{Q}_k := \vec{Q}_k - \lambda \vec{Q}_i$
    $\vec{Q}_i := z \vec{Q}_i$
$p := 1$
for k from 2 to n do
  if $\deg(\vec{Q}_k) < \deg(\vec{Q}_p)$ then $p := k$
output: $\vec{Q}_p$

4 Finding relations with polynomial coefficients

Suppose that $f_1, \ldots, f_n$ are dependent over $k(z)$, let's say $\langle \vec{P}, \vec{f} \rangle = 0$ for some $\vec{P} \in k[z]^n$. The described algorithm is also very useful for finding a relation. Define for every j a finite dimensional vector space $W_j \subseteq V_j$ by

$$W_j = \{\vec{Q} \in V \mid v(\langle \vec{Q}, \vec{f} \rangle) \geq j \wedge \deg(\vec{Q}) \leq \deg(\vec{P})\}$$

Further define $W_\infty = \bigcap_{j=0}^{\infty} W_j$. Now W_∞ contains all relations of $\vec{f}$ of degree $\leq \deg(\vec{P})$, so all $\vec{Q} \in k[z]^n$ satisfying $\langle \vec{Q}, \vec{f} \rangle = 0$ and $\deg(\vec{Q}) \leq \deg(\vec{P})$. We have an infinite descending chain of finite dimensional vector spaces:

$$W_0 \supseteq W_1 \supseteq W_2 \supseteq \ldots$$

So there must be an positive integer N such that $W_\infty = W_N = W_{N+1} = W_{N+2} = \ldots$. If we take d such that $dn + n - 1 \geq N$, then the output of the algorithm will be a $k(z)$-linear relation between $f_1, f_2, \ldots, f_n$.

5 Padé-Hermite forms of arbitrary type

In order to compute Padé-Hermite forms of arbitrary type we make the following definitions: If $d = (d_1, d_2, \ldots, d_n)$ is a vector of non-negative integers and $\vec{P} = (P_1, P_2, \ldots, P_n) \in k[z]^n$ is a polynomial vector then

$$\deg_d(\vec{P}) = \max\{\deg(P_1) - d_1, \deg(P_2) - d_2, \ldots, \deg(P_n) - d_n\}$$

$$\text{and } \mathrm{type}_d(\vec{P}) = \min\{i \mid \deg(P_i) - d_i = \deg_d(P)\}$$

If we replace deg by $\deg_d$ and type by type_d in the algorithm of section 3 then the output will be a Padé- form for $\vec{f}$ of type $(d_1, d_2, \ldots, d_n)$.

References

[1] B. Beckermann, G. Labahn, A uniform approach for Hermite Padé Approximants and their Matrix-type generalizations, Manuscript.

[2] B. Beckermann, G. Labahn, A uniform approach for the fast computation of Matrix-type Padé approximants, Manuscript.

[3] S. Cabay and D. K. Choi, Algebraic computations of scaled padé fractions, Siam J. Comput., Vol. 15, No. 1, pp. 243-270, February 1986.

[4] S. Cabay and G. Labahn, A Fast, Reliable Algorithm for Calculating Padé-Hermite Forms, Proceedings of the ACM-SIGSAM 1989, International Symposium on Symbolic and Algebraic Computations, ACM Press, 1989.

[5] S. Cabay and G. Labahn, Matrix Padé fractions and their computation, Siam J. Comput., Vol 19, No. 4, pp. 639-657, August 1989.

[6] J. Della Dora and C. Dicrescenzo, Approximants de Padé-Hermite, Numer. Math, 43, pp. 23-57, 1984.

[7] H. Padé, Sur la Representation Approchee d'une Fonction par des Fractions Rationelles, Ann Ecole Nor., pp 1-93, supplement Thesis, 1892.

[8] S. Paszkowski, Recurrence Relations in Padé-Hermite Approximation, J. of Comp. and Appl. Math, 19, pp. 99-107, 1987.